On the Complexity of Optimization : Curved Spaces and Benign Landscapes

PhD Oral Exam

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Advisor: Nicolas Boumal

EPFL

OPTIM, Chair of Continuous Optimization

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Outline

Motivation

Part I: Geodesic convexity

Part II: Benign Landscapes

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Part II: Benign Landscapes

Based on:

- "Negative curvature obstructs acceleration for strongly geodesically convex optimization" - C & **Boumal** – COLT'22
- "Curvature and Complexity: Better lower bounds for geodesically convex optimization" - C & **Boumal** – COLT'23
- "Synchronization on circles and spheres with nonlinear interactions" - C, **Rebjock, McRae, Boumal** - under review
- "The sensor network localization problem has benign landscape under mild rank relaxation" - C, **Rebjock, McRae, Boumal** – not yet public

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$$\min_{x \in \mathcal{M}} f(x)$$

Optimization is fundamental in many areas

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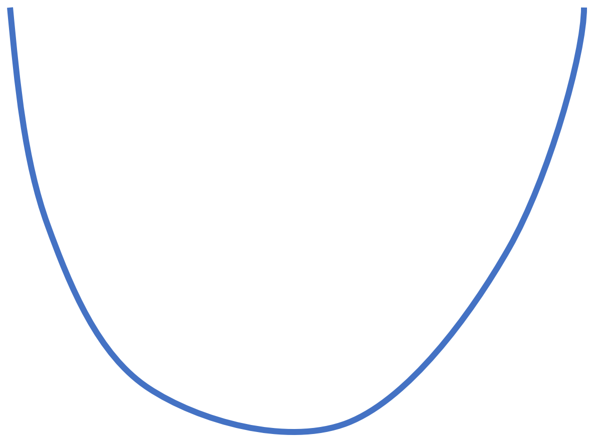

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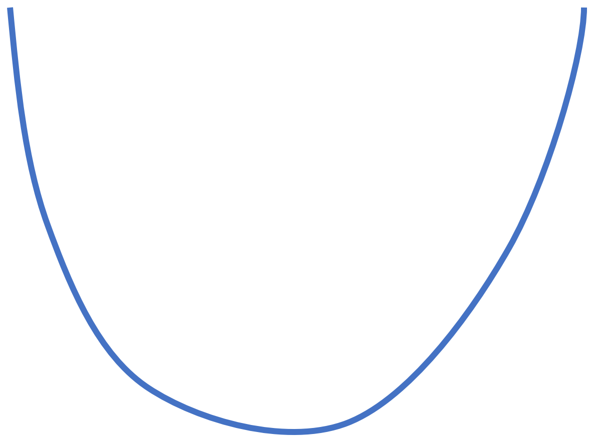

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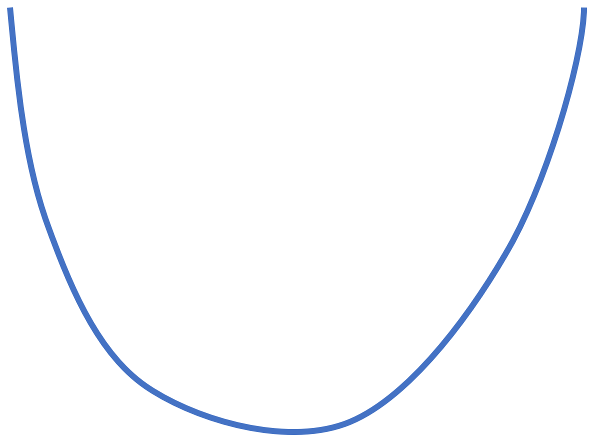

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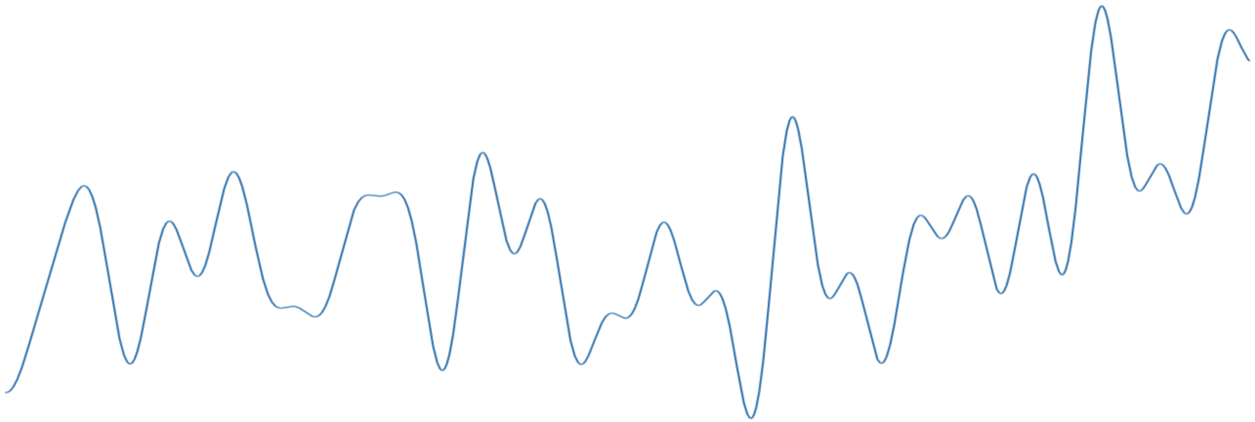

General nonconvex optimization is hard!

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Thesis focuses on 2 special structures:

I. Geodesic convexity – Q2 and Q3

II. Benign landscapes – Q1 (and Q2)

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MLE, aka Tyler's M-estimator:

$$\hat{\Sigma} = \underset{\Sigma \succ 0,\ \det(\Sigma)=1}{\operatorname{argmin}} \frac{d}{m} \sum_{i=1}^{m} \log\left(x_i^\top \Sigma^{-1} x_i\right)$$

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Nonconvex!

But geodesically convex on a Hadamard manifold!

Sources: Tyler, Weisel & Zhang, Franks & Moitra, etc.

Geodesically convex optimization

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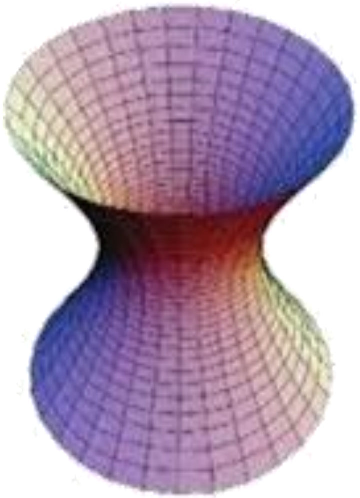

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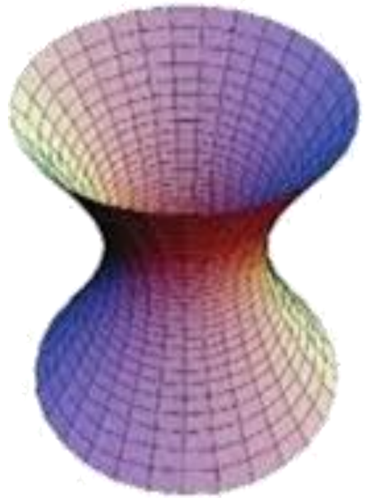

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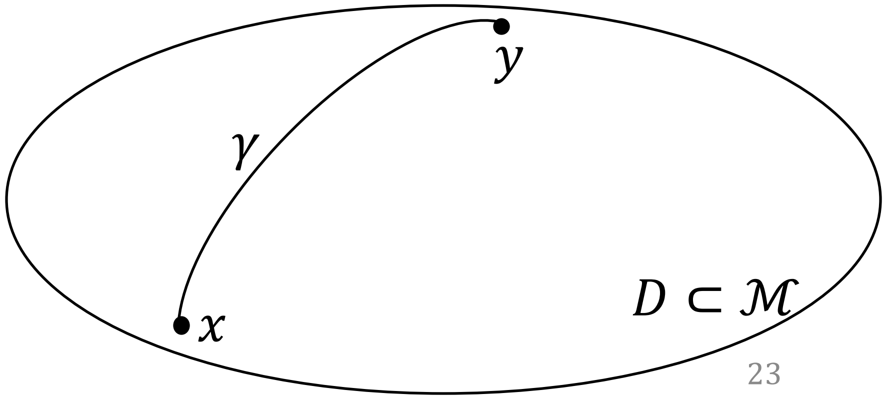
For each $x, y \in D$, there is a unique minimizing geodesic $t \mapsto \gamma(t)$ contained in D, connecting x, y.



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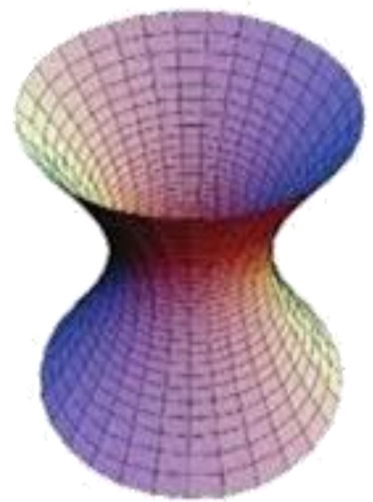

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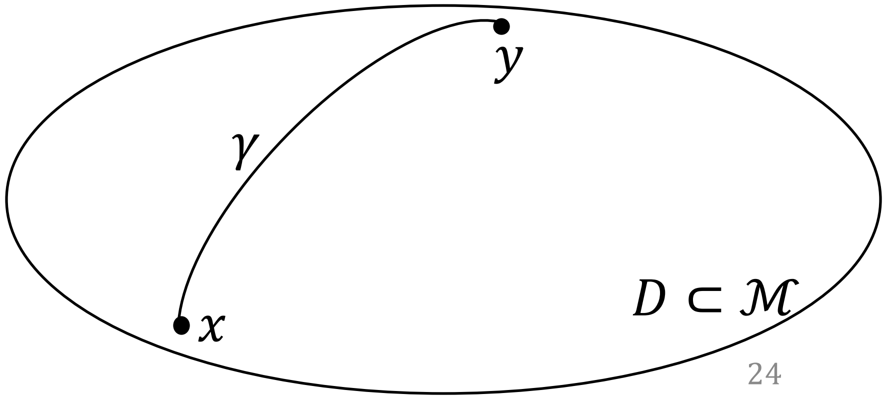

Cost f is μ-strongly g-convex:

$$t \mapsto f(\gamma(t))$$

is μ-strongly convex for any geodesic γ in D.

Complexity of g-convex optim

Our focus: What is the query complexity of g-convex optimization?

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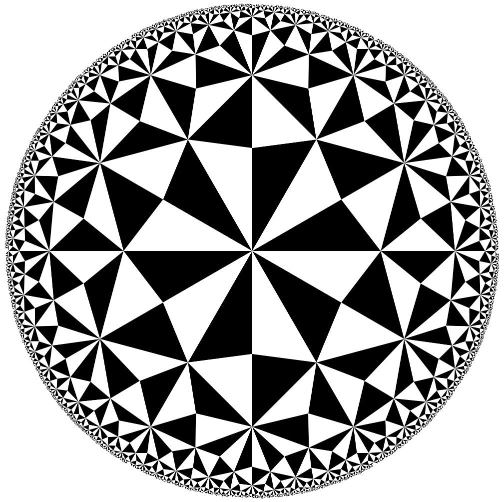

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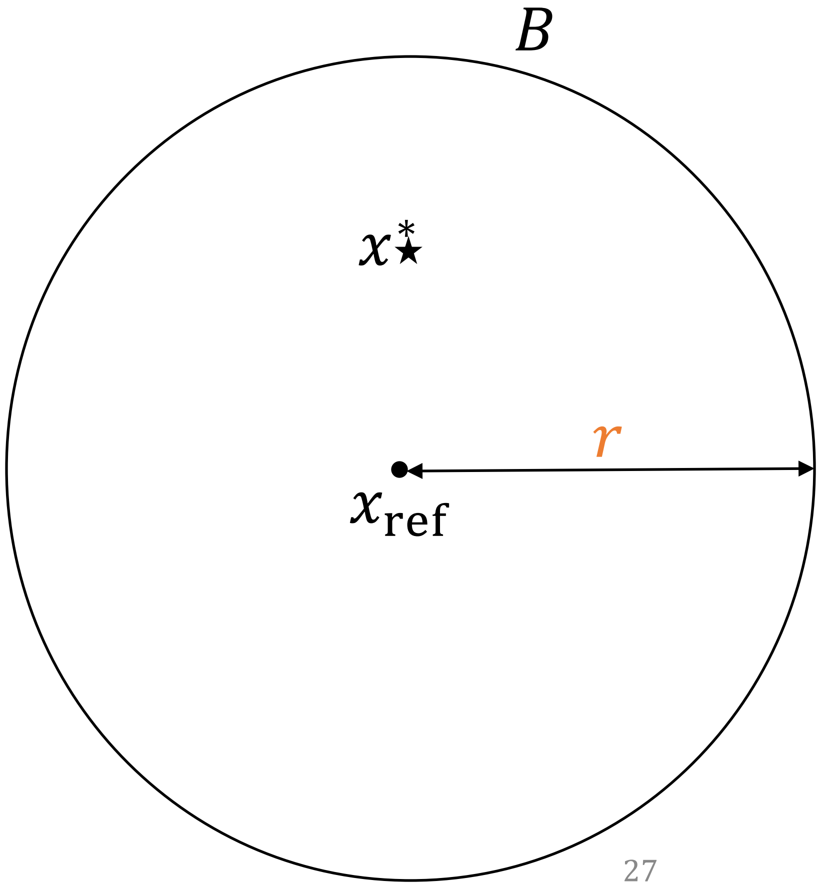

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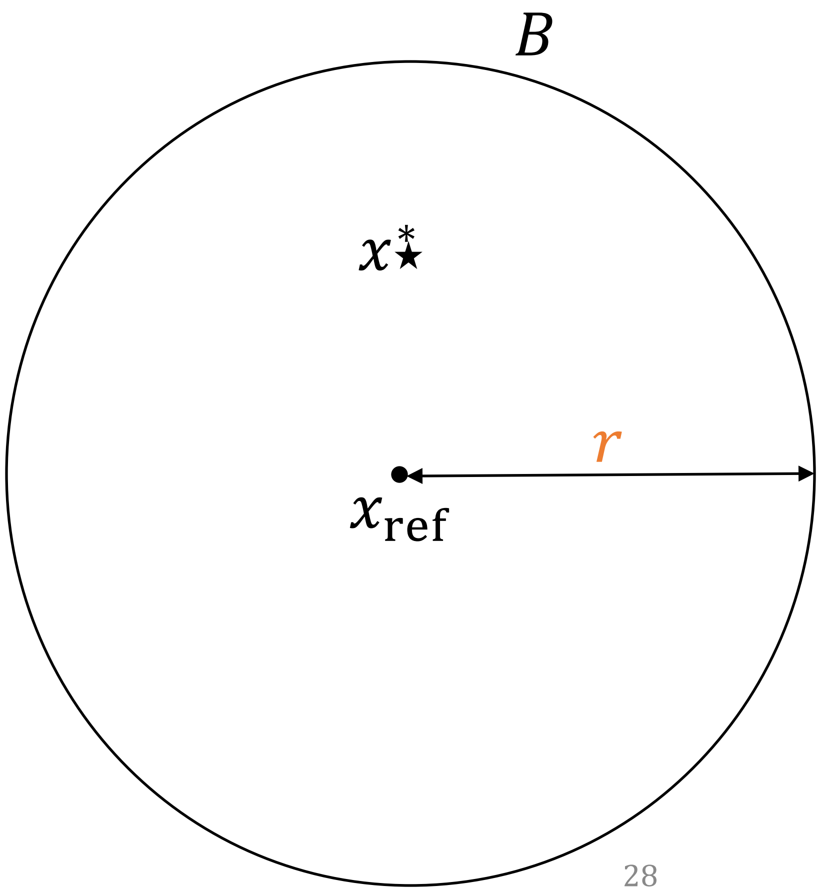

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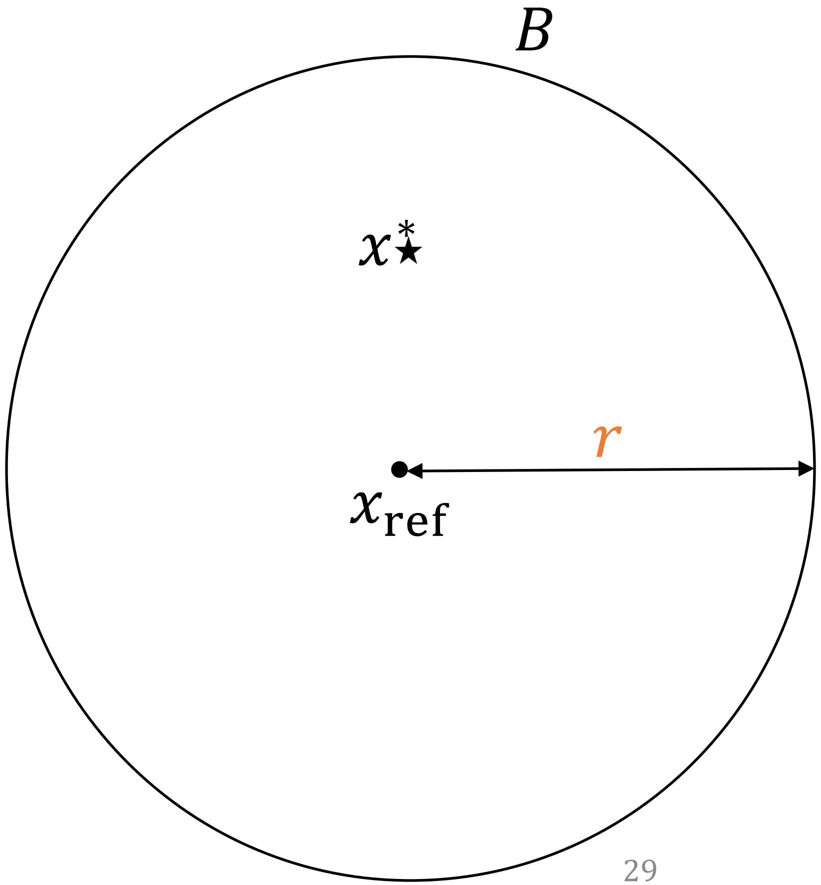

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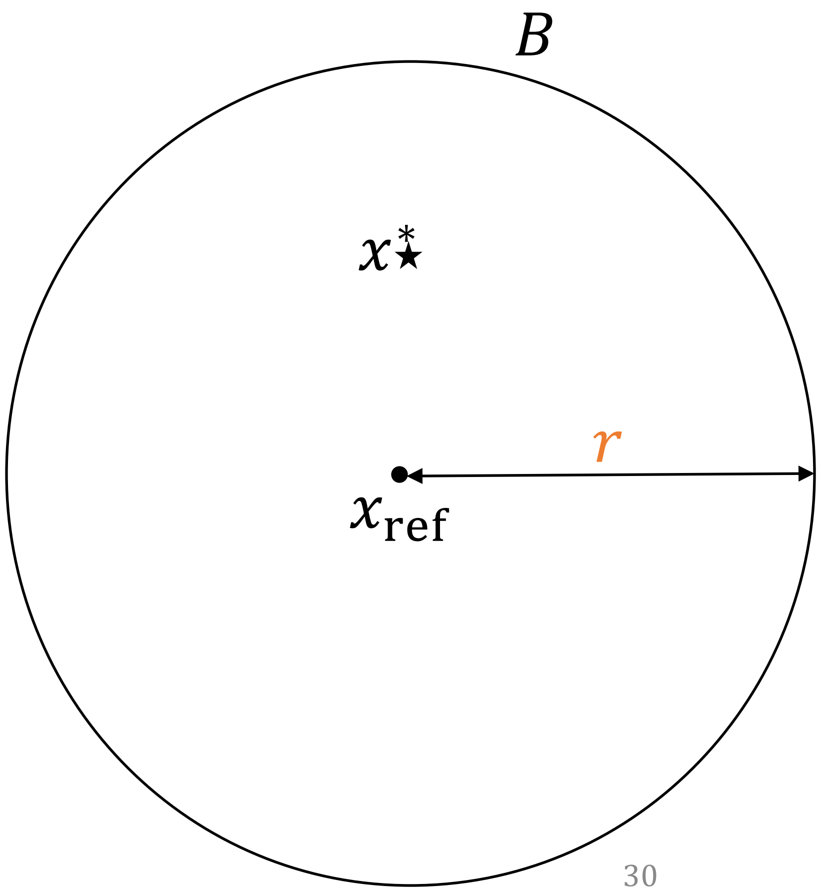
Task: find a point x with accuracy ϵ:

$$f(x) - f^* \leq \epsilon$$

Least number of oracle queries necessary?



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Complexity (i.e., min # of oracle queries) depends on two parameters:

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	$K = 0$	$K < 0$
Constant curvature:	Euclidean space $\mathbb{R}^d$	Hyperbolic space $\mathbb{H}^d$ $(K = -1)$
Volume of ball of radius r:	$\mathrm{Vol} \sim r^d = e^{d\log(r)}$	$\mathrm{Vol} \sim e^{dr}$

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In negatively curved spaces, geodesics diverge rapidly, and so a small error can blow up quickly

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Is there a Riem acceleration scheme with complexity $O\left(\frac{1}{\sqrt{\epsilon}}\right)$? No

Zhang & Sra; Ahn & Sra; Alimisis, Orvieto, Lucchi; Martinez-Rubio; Kim & Yang; etc.

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Bürgisser, Franks, Garg, Oliveira, Walter, Wigderson '21

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   - For tensor scaling (and others), radius r is exponentially large in input size (Franks & Reichenbach'21) – so lower bound points to difficulty of finding poly-time algos

Bürgisser, Franks, Garg, Oliveira, Walter, Wigderson '21

Part I: Geodesic convexity

Much more in thesis!

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- Bounds depending on both ϵ and ζ.
- Other problem classes.

g-convex setting	Lower bound	Upper bound	Algorithm
(P1) Lipschitz, lo-dim	$\tilde{\Omega}(\zeta + d)$	$O(\zeta d)$	Center of gravity (Criscitiello, et al. '23)
(P2) Lipschitz	$\tilde{\Omega}\left(\zeta + \frac{1}{\zeta^2 \epsilon^2}\right)$	$O\left(\frac{\zeta}{\epsilon^2}\right)$	Subgradient descent (Zhang & Sra'16)
(P3) Smooth	$\tilde{\Omega}\left(\zeta + \frac{1}{\zeta\sqrt{\epsilon}}\right)$	$\tilde{O}\left(\sqrt{\zeta/\epsilon}\right)$	RNAG-C (Kim & Yang'22 / Martinez-Rubio et al.'22)
(P4) Smooth, strongly g-convex	$\tilde{\Omega}(\zeta + \sqrt{\kappa})$	$\tilde{O}(\sqrt{\zeta\kappa})$	RNAG-SC (Kim & Yang'22 / Martinez-Rubio et al.'22)

Cutting-planes game	$\tilde{\Omega}(\zeta d)$	$O(\zeta d)$	Center of gravity (Criscitiello, et al. '23)

$\Omega\left(\frac{\zeta}{\epsilon^2}\right)$ lower bound for subgradient descent (with Polyak step size)

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Open question:

Matching upper and lower bounds?

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Benign landscape

$$\min_{x \in \mathbb{R}^d} f(x)$$

Some problems are g-convex, but many are not!

Need a more general **structure!**

Benign landscape

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Definition: f has a **benign landscape** if all 2-critical points are optimal:

$\nabla f(x) = 0$ and $\nabla^2 f(x) \succcurlyeq 0$ implies x is a global min

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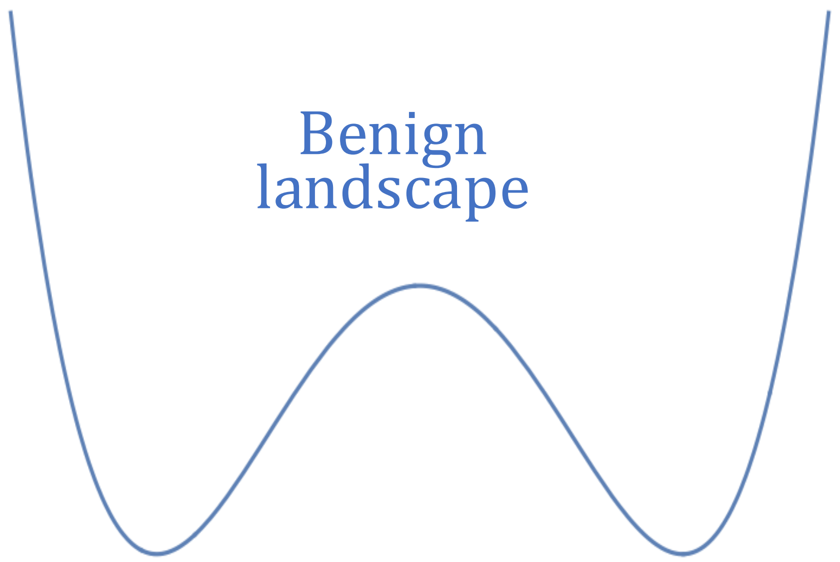

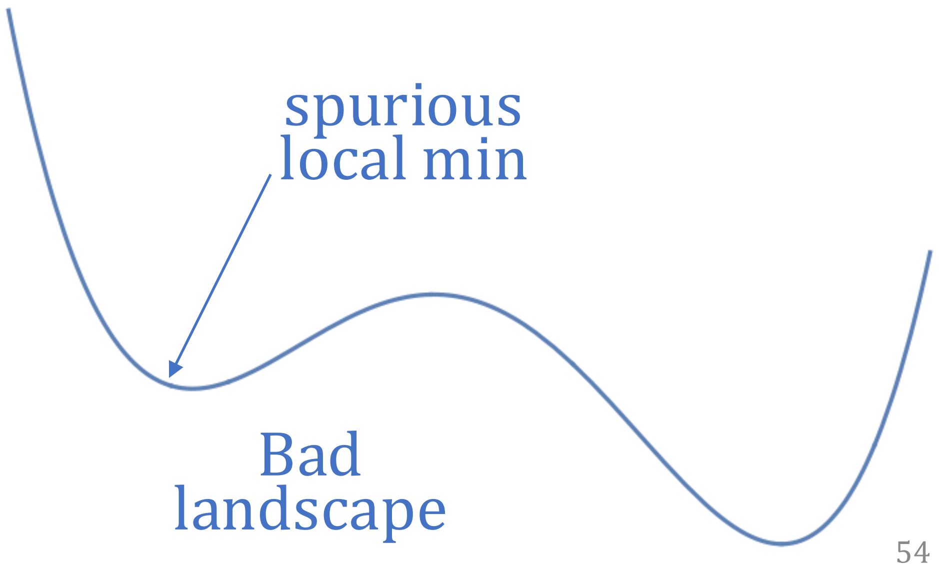

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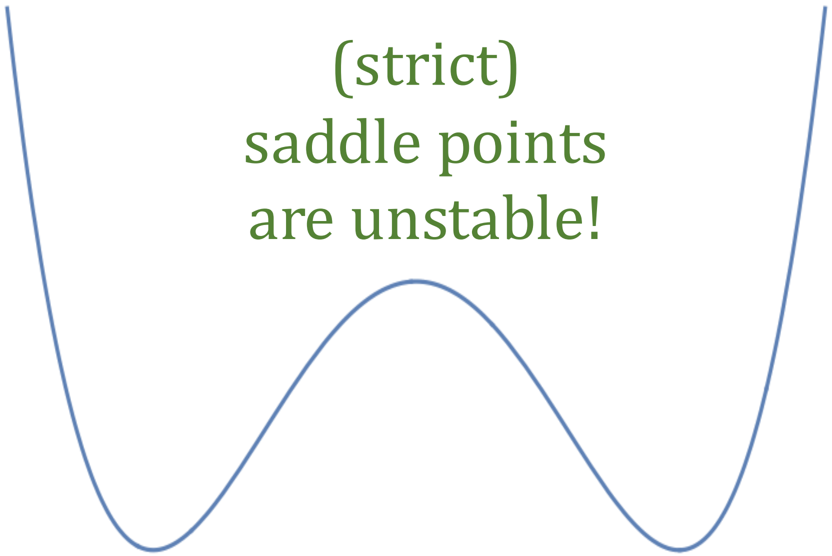

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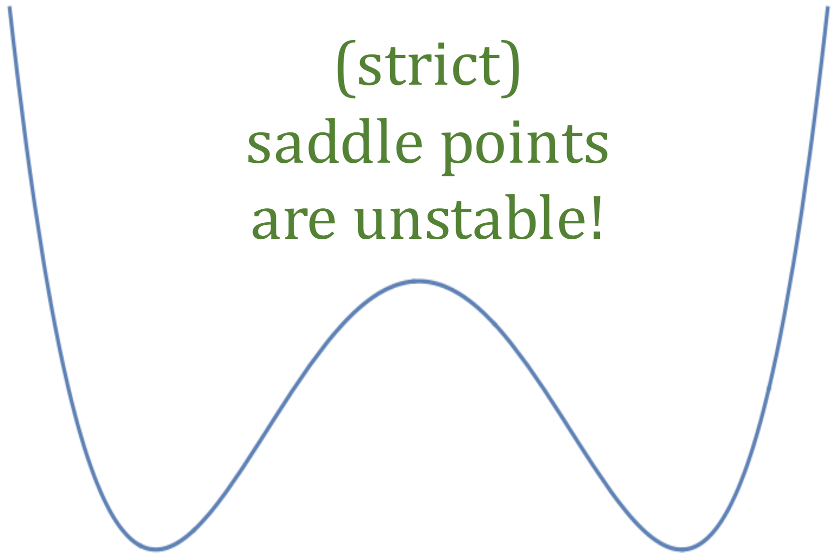

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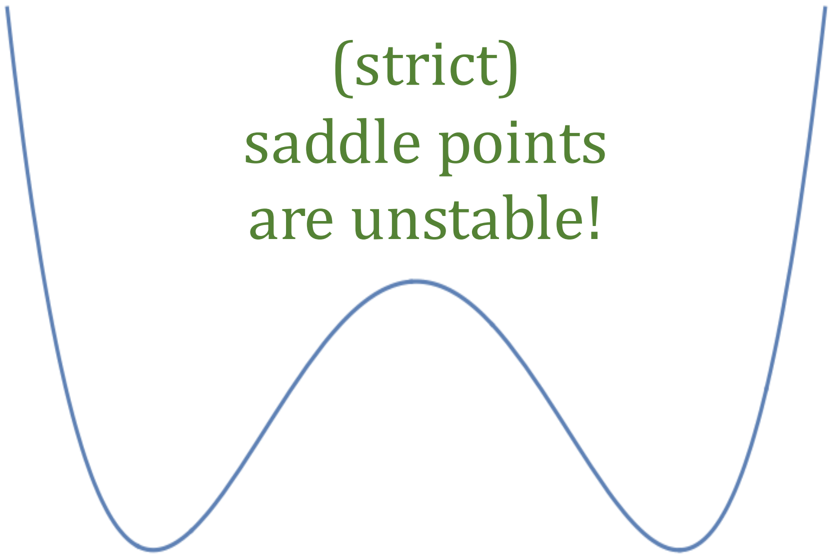
+
Łojasiewicz theorem



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saddle points
are unstable!

Benign landscape

$$\min_{x \in \mathbb{R}^d} f(x)$$

Definition: f has a **benign landscape** if all 2-critical points are optimal:

$\nabla f(x) = 0$ and $\nabla^2 f(x) \succcurlyeq 0$ implies x is a global min

Part II of thesis:
Understand when several problems have benign landscape. With an eye towards new tools.

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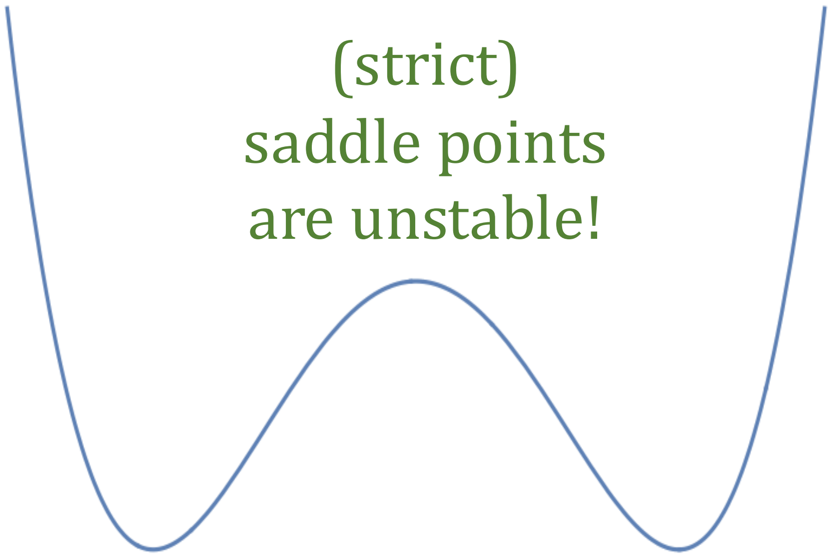

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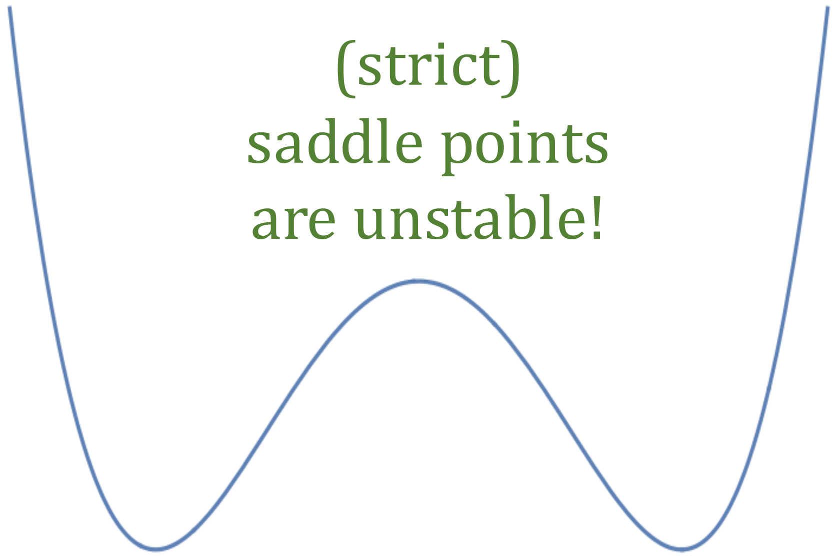
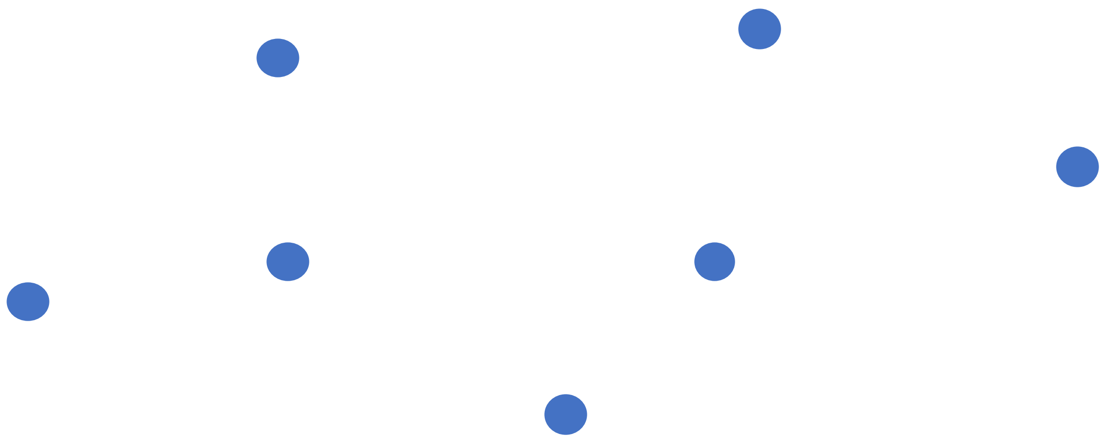
Our focus next: sensor network localization



The problem: SNL

n unknown points $z_1^*, z_2^*, \dots, z_n^*$ in $\mathbb{R}^\ell$.



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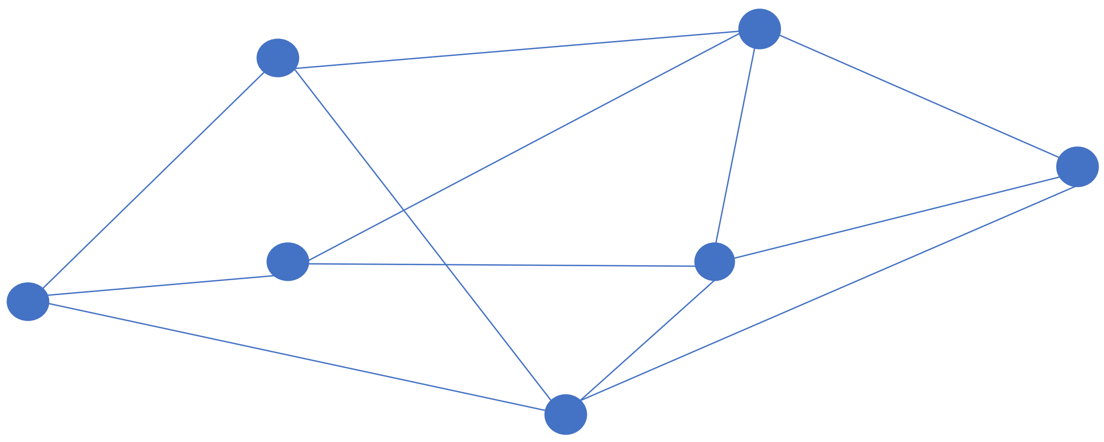

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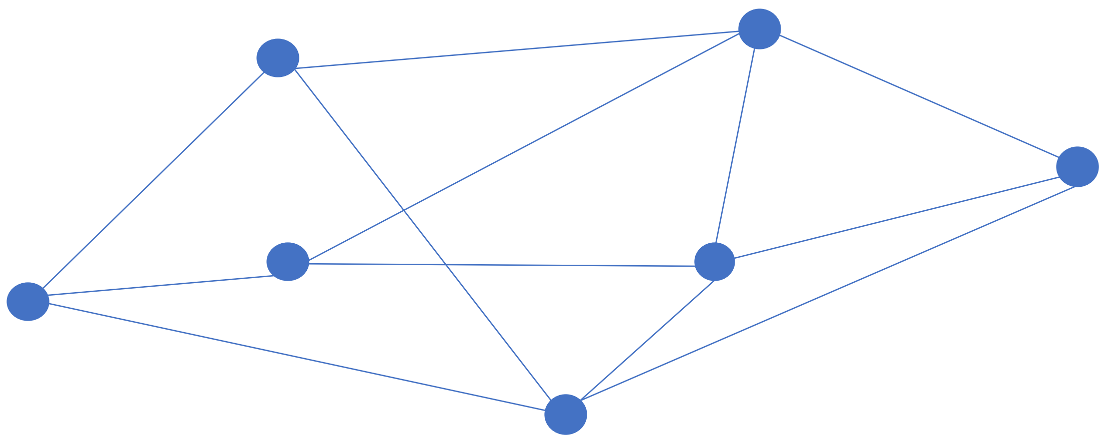

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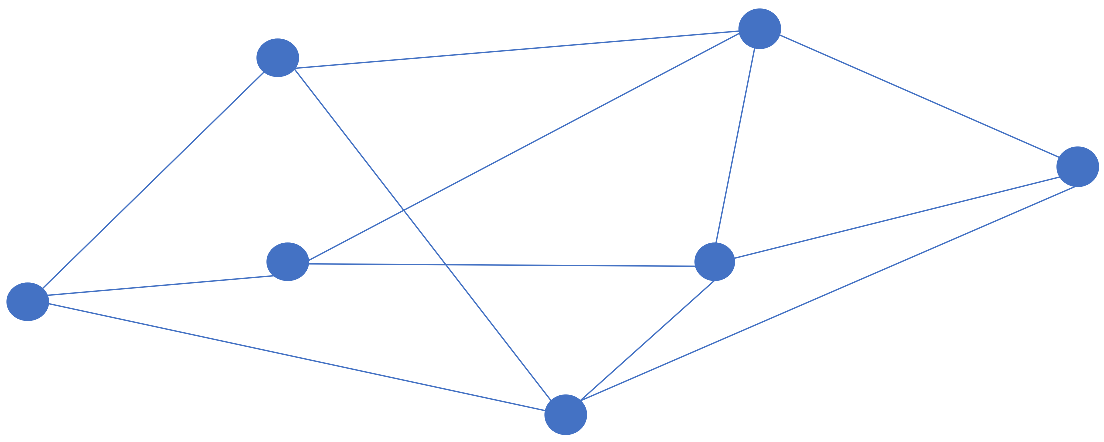

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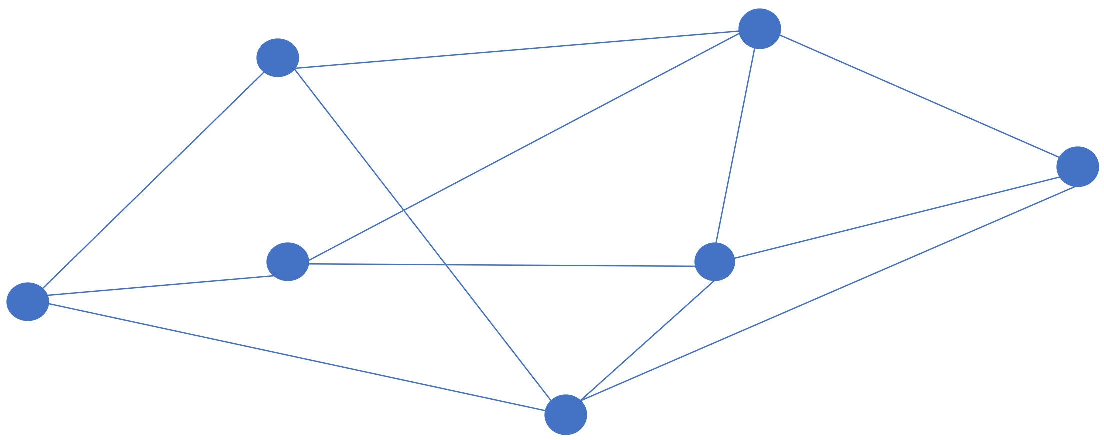

Applications in robotics, determining molecular conformations, data analysis

Optimization problem

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Nonconvex!

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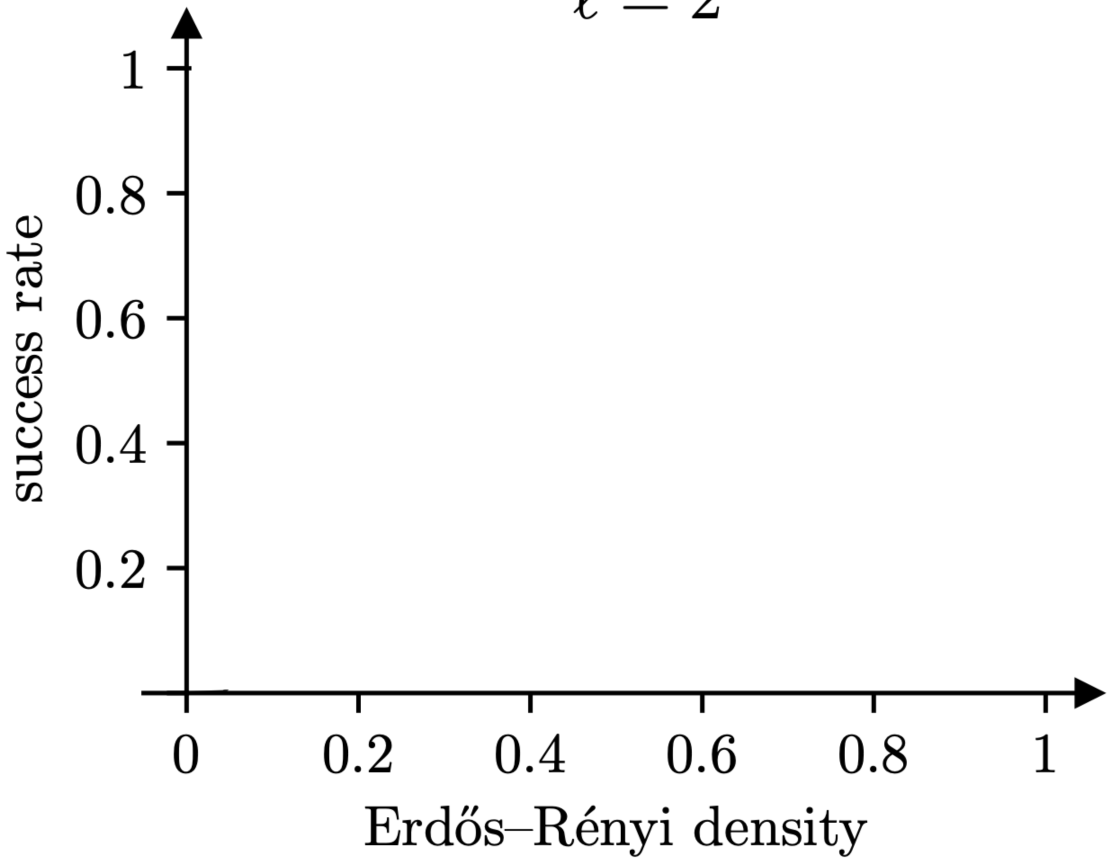
- Graph = ER
- Run TR from random initialization

$\ell = 2$



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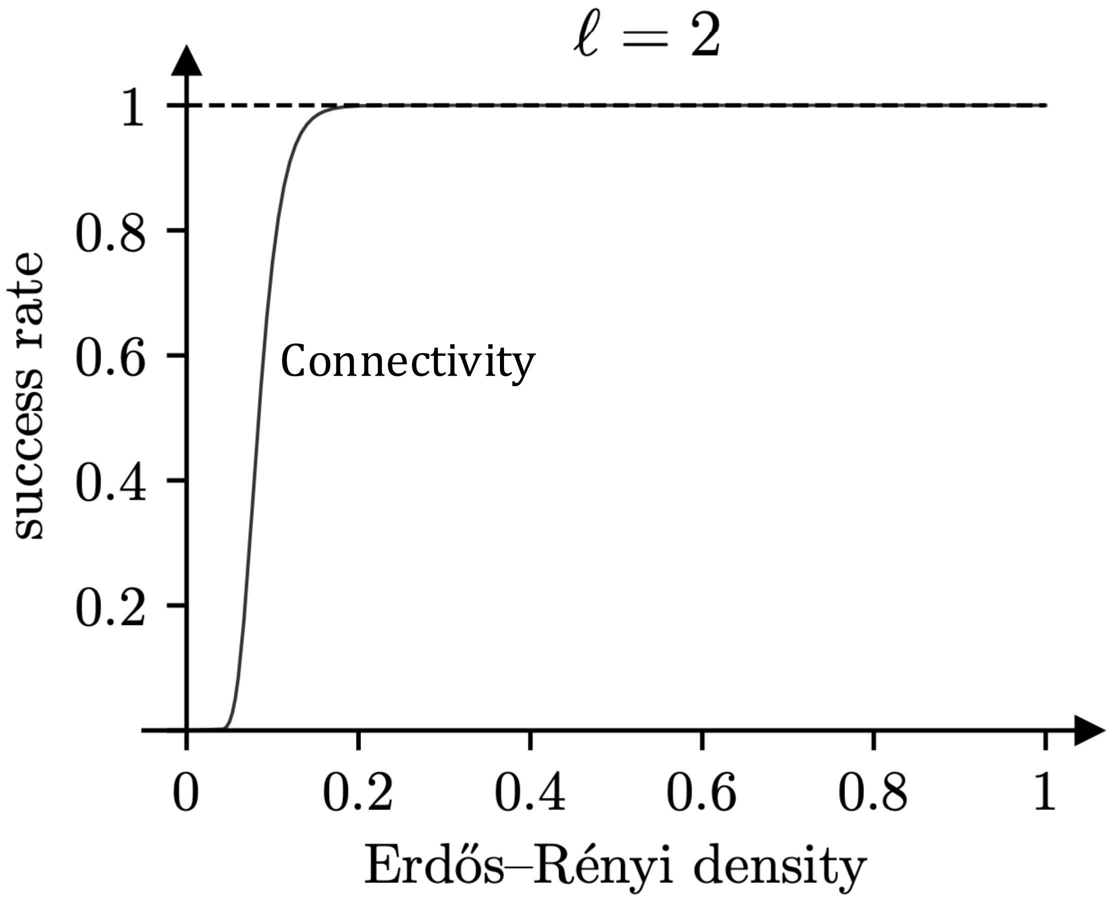

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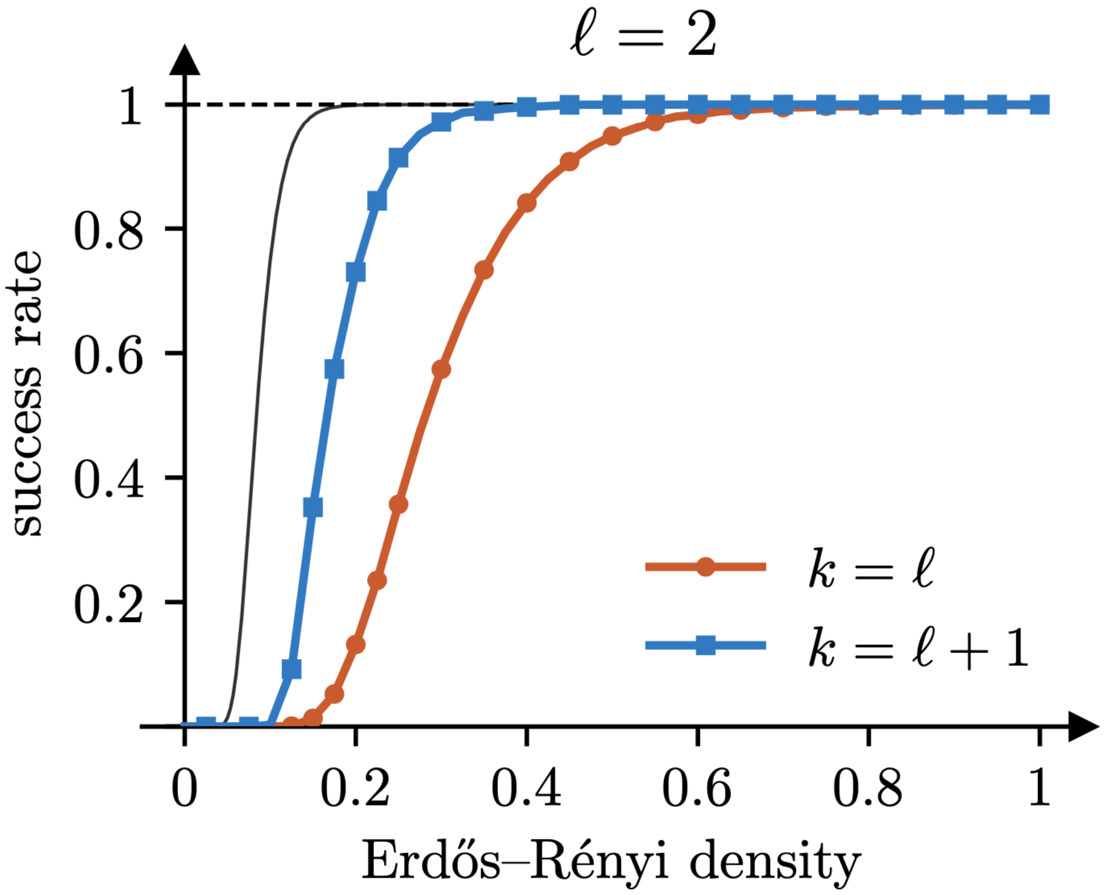

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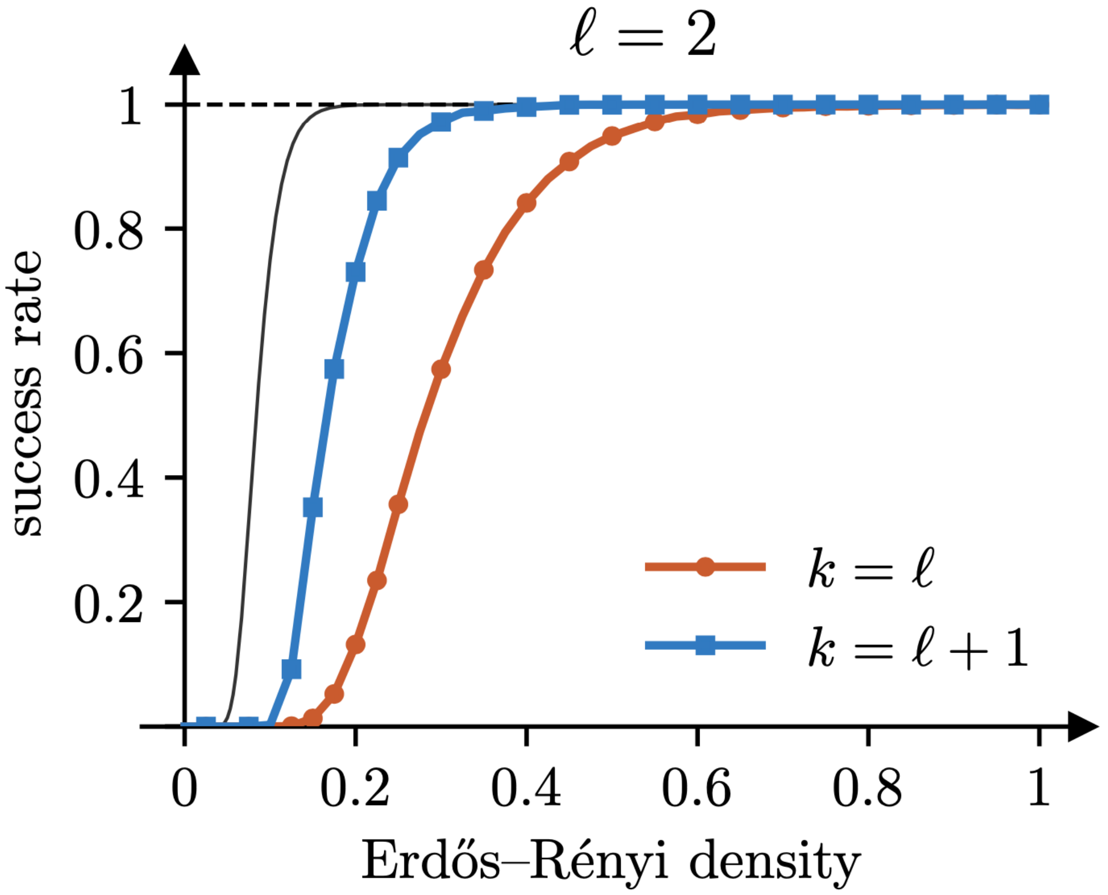
Theoretical explanation?



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Our focus: (nearly) all distances known.
Landscape for s-stress and its relaxations unknown even in this simplest of cases!

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"s-stress"

Results

Theorem [relaxation is necessary]: If $k = \ell$ (no relaxation), landscape may be non-benign even if graph is complete.

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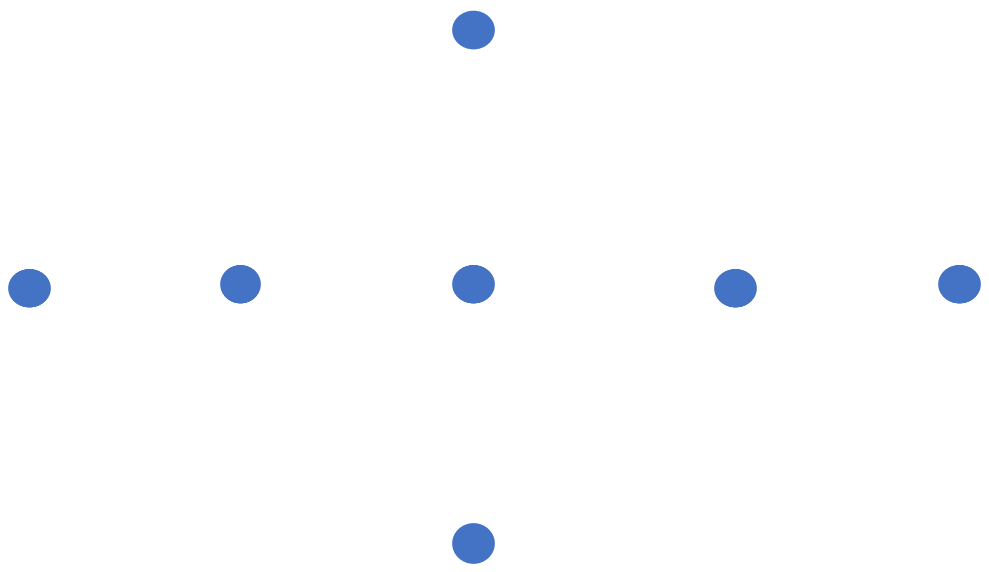
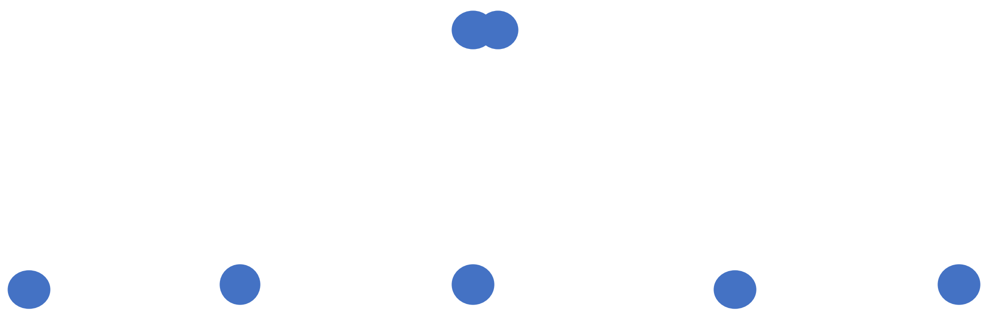
s-stress can have spurious strict local minima, with complete graph

Ground truth $z_1^*, z_2^*, \ldots$



Spurious configuration $z_1, z_2, \ldots$



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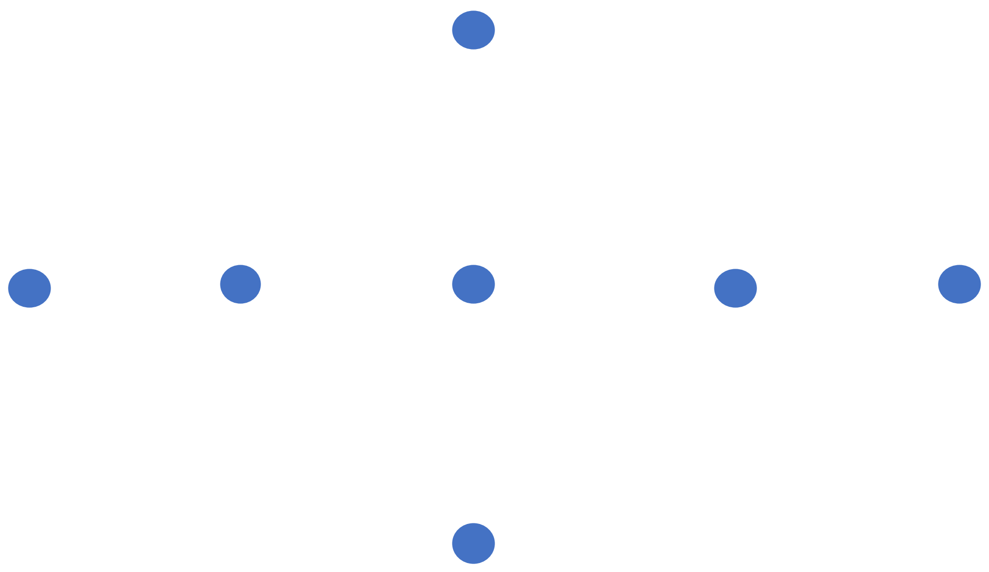

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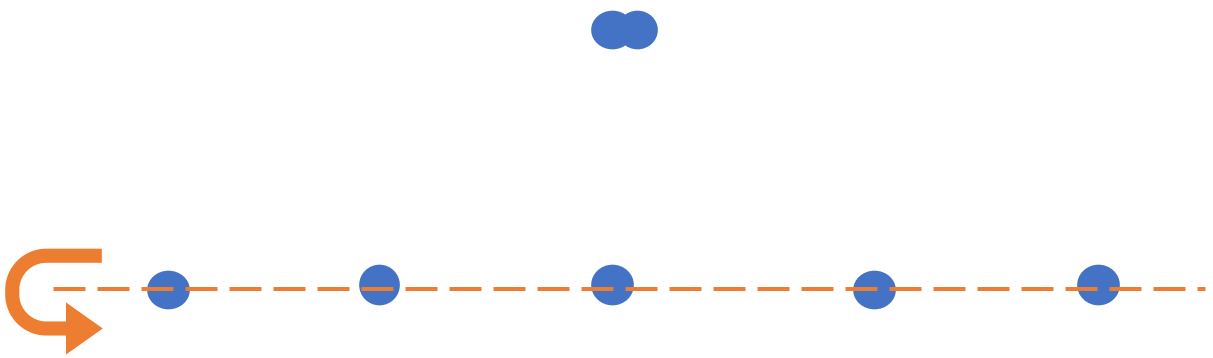

Optimize over points in $\mathbb{R}^3$

Results

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Going forward: tools for proving benign landscapes?